DIFFERENTIAL EQUATIONS

1 Homogeneous differential equations of order one

Consider the following order 1 homogeneous differential equation:

$$(E_0) \quad a(t)y'(t) + b(t)y(t) = 0$$

where a and b are two functions, defined on an interval $J \subset \mathbb{R}$.

Theorem (Solutions set of (E_0)).

If there exists an interval $I \subset J$ on which the functions a and b are continuous and where the function a is always non-zero valued, then the solution set of (E_0) on I is:

$$S_0 = \left\{ \begin{array}{ccc} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{-\int \frac{b(t)}{a(t)}dt} \end{array} ; k \in \mathbb{R} \right\}$$

General method: our purpose is to prove an equality between two sets, the set S_0 and the right hand side set (which will be denoted by B in the following). We hence prove two inclusions: $S_0 \subset B$ and $B \subset S_0$.
For example, proving that $S_0 \subset B$ requires proving that every element of S_0 is also an element of B.

Proof:

In the whole proof, $F(t) = \int \frac{b(t)}{a(t)}\mathrm{d}t$ will denote a primitive function (antiderivative) of $\frac{b}{a}$ on I. (a primitive function exists since the function $\frac{b}{a}$ is defined and continuous on I).
Thus: $\forall\, t \in I,\ F'(t) = \dfrac{b(t)}{a(t)}$.

$\boxed{\subset}$ Let $y_0 \in S_0$ be a solution of (E_0) on I: $\forall\, t \in I,\ a(t)y_0'(t) + b(t)y_0(t) = 0$.

Consider the function $z : t \longmapsto y_0(t)e^{F(t)}$. This function z is defined and differentiable on I.
Furthermore, for all $t \in I$,

$$\begin{aligned} z'(t) &= y_0'(t)e^{F(t)} + y_0(t)F'(t)e^{F(t)} \\ &= y_0'(t)e^{F(t)} + y_0(t)\frac{b(t)}{a(t)}e^{F(t)} \\ &= \frac{e^{F(t)}}{a(t)}\Big(a(t)y_0'(t) + b(t)y_0(t)\Big) \\ &= \frac{e^{F(t)}}{a(t)} \times 0 = 0 \end{aligned}$$

Hence, z is a constant function on I. Let $k \in \mathbb{R}$ be this constant value on I.

Then we get: $\forall\, t \in I,\ z(t) = y_0(t)e^{F(t)} = k$, that is, $y_0(t) = ke^{-F(t)} = ke^{-\int \frac{b(t)}{a(t)}\mathrm{d}t}$.

$\boxed{\supset}$ Let a function y be defined on I by $y(t) = ke^{-F(t)}$ where $k \in \mathbb{R}$. This function y is differentiable on I and

$$\forall\, t \in I,\ y'(t) = -kF'(t)e^{-F(t)} = -F'(t)y(t) = -\frac{b(t)}{a(t)}y(t)$$

Thus, $\forall\, t \in I,\ a(t)y'(t) + b(t)y(t) = a(t)\left(-\frac{b(t)}{a(t)}y(t)\right) + b(t)y(t) = -b(t)y(t) + b(t)y(t) = 0$

That is, $y \in S_0$.

2 Solution set of a differential equation of order one

Theorem (Solutions set of (E)).

Consider the equations (E) $a(t)y'(t) + b(t)y(t) = c(t)$ and (E_0) $a(t)y'(t) + b(t)y(t) = 0$
where a, b and c are three continuous functions on an interval $I \subset \mathbb{R}$ such that $\forall\, t \in I,\ a(t) \neq 0$.

Let y_p be a particular solution of (E).

Then:

$$S = \{y_p + y_0;\ y_0 \in S_0\}$$

where S_0 is the solution set of (E_0).

Proof:

$\boxed{\subset}$ Let $y \in S$ be a solution of (E), that is: $\forall\, t \in I,\ a(t)y'(t) + b(t)y(t) = c(t)$.

Consider the function y_0 defined for all $t \in I$ by $y_0(t) = y(t) - y_p(t)$. Then y_0 is differentiable on I and for all $t \in I$, $y_0'(t) = y'(t) - y_p'(t)$. Thus, for all $t \in I$,

$$\begin{aligned} a(t)y_0'(t) + b(t)y_0(t) &= a(t)\Big(y'(t) - y_p'(t)\Big) + b(t)\Big(y(t) - y_p(t)\Big) \\ &= \Big(a(t)y'(t) + b(t)y(t)\Big) - \Big(a(t)y_p'(t) + b(t)y_p(t)\Big) \\ &= c(t) - c(t) = 0 \end{aligned}$$

This proves that $y_0 \in S_0$. Thus, there exists $y_0 \in S_0$ such that $y = y_0 + y_p$.

$\boxed{\supset}$ Let $f \in \{y_p + y_0;\ y_0 \in S_0\}$. Then there exists $y_0 \in S_0$ such that $f = y_p + y_0$: for all $t \in I$, $f(t) = y_p(t) + y_0(t)$.

Thus, f is differentiable on I and for all $t \in I$, $f'(t) = y_p'(t) + y_0'(t)$.

We deduce that for all $t \in I$,:

$$\begin{aligned} a(t)f'(t) + b(t)f(t) &= a(t)\Big(y_p'(t) + y_0'(t)\Big) + b(t)\Big(y_p(t) + y_0(t)\Big) \\ &= \Big(a(t)y_p'(t) + b(t)y_p(t)\Big) + \Big(a(t)y_0'(t) + b(t)y_0(t)\Big) \\ &= c(t) + 0 = c(t) \end{aligned}$$

Thus, $f \in S$.